

BOURGAIN-BREZIS-MIRONESCU FORMULA FOR MAGNETIC OPERATORS

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ABSTRACT. We prove a Bourgain-Brezis-Mironescu type formula for a class of nonlocal magnetic spaces, which builds a bridge between a fractional magnetic operator recently introduced and the classical theory.

1. INTRODUCTION

Let $s \in (0, 1)$ and $N > 2s$. If $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a smooth function, the nonlocal operator

$$(-\Delta)_A^s u(x) = c(N, s) \lim_{\varepsilon \searrow 0} \int_{B_\varepsilon^c(x)} \frac{u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)}{|x - y|^{N+2s}} dy, \quad x \in \mathbb{R}^N,$$

has been recently introduced in [6], where the ground state solutions of $(-\Delta)_A^s u + u = |u|^{p-2}u$ in the three dimensional setting have been obtained via concentration compactness arguments. If $A = 0$, then the above operator is consistent with the usual notion of fractional Laplacian. The motivations that led to its introduction are carefully described in [6] and rely essentially on the Lévy-Khintchine formula for the generator of a general Lévy process. We point out that the normalization constant $c(N, s)$ satisfies

$$\lim_{s \nearrow 1} \frac{c(N, s)}{1 - s} = \frac{4N\Gamma(N/2)}{2\pi^{N/2}},$$

where Γ denotes the Gamma function. For the sake of completeness, we recall that different definitions of nonlocal magnetic operator are viable, see e.g. [8, 9]. All these notions aim to extend the well-know definition of the magnetic Schrödinger operator

$$-(\nabla - iA(x))^2 u = -\Delta u + 2iA(x) \cdot \nabla u + |A(x)|^2 u + iu \operatorname{div} A(x),$$

namely the differential of the energy functional

$$\mathcal{E}_A(u) = \int_{\mathbb{R}^N} |\nabla u - iA(x)u|^2 dx,$$

for which we refer the reader to [1, 2, 11] and the included references. In order to corroborate the justification for the introduction of $(-\Delta)_A^s$, in this note we prove that a well-known formula due to Bourgain, Brezis and Mironescu (see [3, 4, 10]) for the limit of the Gagliardo semi-norm of $H^s(\Omega)$ as $s \nearrow 1$ extends to the magnetic setting. As a consequence, in a suitable sense, from the nonlocal to the local regime, it holds

$$(-\Delta)_A^s u \rightsquigarrow (\nabla - iA(x))^2 u, \quad \text{for } s \nearrow 1.$$

We consider

$$[u]_{H_A^1(\Omega)} := \sqrt{\int_{\Omega} |\nabla u - iA(x)u|^2 dx},$$

and define $H_A^1(\Omega)$ as the space of functions $u \in L^2(\Omega, \mathbb{C})$ such that $[u]_{H_A^1(\Omega)} < \infty$ endowed with the norm

$$\|u\|_{H_A^1(\Omega)} := \sqrt{\|u\|_{L^2(\Omega)}^2 + [u]_{H_A^1(\Omega)}^2}.$$

Our main results are the following

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Theorem 1.1 (Magnetic Bourgain-Brezis-Mironescu). *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary and $A \in C^2(\bar{\Omega})$. Then, for every $u \in H_A^1(\Omega)$, we have*

$$\lim_{s \nearrow 1} (1-s) \int_{\Omega} \int_{\Omega} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^2}{|x-y|^{N+2s}} dx dy = K_N \int_{\Omega} |\nabla u - iA(x)u|^2 dx,$$

where

$$(1.1) \quad K_N = \frac{1}{2} \int_{\mathbb{S}^{N-1}} |\omega \cdot \mathbf{e}|^2 d\mathcal{H}^{N-1}(\omega),$$

being \mathbb{S}^{N-1} the unit sphere and \mathbf{e} any unit vector in \mathbb{R}^N .

As a variant of Theorem 1.1, if $H_{0,A}^1(\Omega)$ denotes the closure of $C_c^\infty(\Omega)$ in $H_A^1(\Omega)$, we get the following

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Assume that $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is locally bounded and $A \in C^2(\bar{\Omega})$. Then, for every $u \in H_{0,A}^1(\Omega)$, we have*

$$\lim_{s \nearrow 1} (1-s) \int_{\mathbb{R}^{2N}} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^2}{|x-y|^{N+2s}} dx dy = K_N \int_{\Omega} |\nabla u - iA(x)u|^2 dx.$$

Notations. Let $\Omega \subset \mathbb{R}^N$ be an open set. We denote by $L^2(\Omega, \mathbb{C})$ the Lebesgue space of complex valued functions with summable square. For $s \in (0, 1)$, the magnetic Gagliardo semi-norm is

$$[u]_{H_A^s(\Omega)} := \sqrt{\int_{\Omega} \int_{\Omega} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^2}{|x-y|^{N+2s}} dx dy}.$$

We denote by $H_A^s(\Omega)$ the space of functions $u \in L^2(\Omega, \mathbb{C})$ such that $[u]_{H_A^s(\Omega)} < \infty$ endowed with

$$\|u\|_{H_A^s(\Omega)} := \sqrt{\|u\|_{L^2(\Omega)}^2 + [u]_{H_A^s(\Omega)}^2}.$$

We denote by $B(x_0, R)$ the ball in \mathbb{R}^N of center x_0 and radius $R > 0$. For any set $E \subset \mathbb{R}^N$ we will denote by E^c the complement of E . For $A, B \subset \mathbb{R}^N$ open and bounded, $A \Subset B$ means $\bar{A} \subset B$.

2. PRELIMINARY RESULTS

We start with the following Lemma.

Lemma 2.1. *Assume that $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is locally bounded. Then, for any compact $V \subset \mathbb{R}^N$ with $\Omega \Subset V$, there exists $C = C(A, V) > 0$ such that*

$$\int_{\mathbb{R}^N} |u(y+h) - e^{ih \cdot A(y+\frac{h}{2})} u(y)|^2 dy \leq C|h|^2 \|u\|_{H_A^1(\mathbb{R}^N)}^2,$$

for all $u \in H_A^1(\mathbb{R}^N)$ such that $u = 0$ on V^c and any $h \in \mathbb{R}^N$ with $|h| \leq 1$.

Proof. Assume first that $u \in C_0^\infty(\mathbb{R}^N)$ with $u = 0$ on V^c . Fix $y, h \in \mathbb{R}^N$ and define

$$\varphi(t) := e^{i(1-t)h \cdot A(y+\frac{h}{2})} u(y+th), \quad t \in [0, 1].$$

Then we have

$$u(y+h) - e^{ih \cdot A(y+\frac{h}{2})} u(y) = \varphi(1) - \varphi(0) = \int_0^1 \varphi'(t) dt,$$

and since

$$\varphi'(t) = e^{i(1-t)h \cdot A(y+\frac{h}{2})} h \cdot \left(\nabla_y u(y+th) - iA\left(y+\frac{h}{2}\right) u(y+th) \right),$$

by Hölder inequality we get

$$|u(y+h) - e^{ih \cdot A(y+\frac{h}{2})} u(y)|^2 \leq |h|^2 \int_0^1 \left| \nabla_y u(y+th) - iA\left(y+\frac{h}{2}\right) u(y+th) \right|^2 dt.$$

Therefore, integrating with respect to y over \mathbb{R}^N and using Fubini's Theorem, we get

$$\begin{aligned} \int_{\mathbb{R}^N} |u(y+h) - e^{ih \cdot A(y+\frac{h}{2})} u(y)|^2 dy &\leq |h|^2 \int_0^1 dt \int_{\mathbb{R}^N} \left| \nabla_y u(y+th) - iA\left(y+\frac{h}{2}\right) u(y+th) \right|^2 dy \\ &= |h|^2 \int_0^1 dt \int_{\mathbb{R}^N} \left| \nabla_z u(z) - iA\left(z+\frac{1-2t}{2}h\right) u(z) \right|^2 dz \\ &\leq 2|h|^2 \int_{\mathbb{R}^N} |\nabla_z u(z) - iA(z) u(z)|^2 dz \\ &\quad + 2|h|^2 \int_V \left| A\left(z+\frac{1-2t}{2}h\right) - A(z) \right|^2 |u(z)|^2 dz. \end{aligned}$$

Then, since A is bounded on the set V , we have for some constant $C > 0$

$$\begin{aligned} \int_{\mathbb{R}^N} |u(y+h) - e^{ih \cdot A(y+\frac{h}{2})} u(y)|^2 dy &\leq C|h|^2 \left(\int_{\mathbb{R}^N} |\nabla_z u(z) - iA(z) u(z)|^2 dz + \int_{\mathbb{R}^N} |u(z)|^2 dz \right) \\ &= C|h|^2 \|u\|_{H_A^1(\mathbb{R}^N)}^2. \end{aligned}$$

When dealing with a general u we can argue by a density argument. \square

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary, $V \subset \mathbb{R}^N$ a compact set with $\Omega \Subset V$ and $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ locally bounded. Then there exists $C(\Omega, V, A) > 0$ such that for any $u \in H_A^1(\Omega)$ there exists $Eu \in H_A^1(\mathbb{R}^N)$ such that $Eu = u$ in Ω , $Eu = 0$ in V^c and*

$$\|Eu\|_{H_A^1(\mathbb{R}^N)} \leq C(\Omega, V, A) \|u\|_{H_A^1(\Omega)}.$$

Proof. Observe that, for any bounded set $W \subset \mathbb{R}^N$ there exist $C_1(A, W), C_2(A, W) > 0$ with

$$C_1(A, W) \|u\|_{H^1(W)} \leq \|u\|_{H_A^1(W)} \leq C_2(A, W) \|u\|_{H^1(W)}, \quad \text{for any } u \in H^1(W).$$

This follows easily, via simple computations, by the definition of the norm of $H_A^1(W)$ and in view of the local boundedness assumption on the potential A . Now, by the standard extension property for $H^1(\Omega)$ (see e.g. [7, Theorem 1, p.254]) there exists $C(\Omega, V) > 0$ such that for any $u \in H^1(\Omega)$ there exists a function $Eu \in H^1(\mathbb{R}^N)$ such that $Eu = u$ in Ω , $Eu = 0$ in V^c and $\|Eu\|_{H^1(\mathbb{R}^N)} \leq C(\Omega, V) \|u\|_{H^1(\Omega)}$. Then, for any $u \in H_A^1(\Omega)$, we get

$$\begin{aligned} \|Eu\|_{H_A^1(\mathbb{R}^N)} &= \|Eu\|_{H_A^1(V)} \leq C_2(A, V) \|Eu\|_{H^1(V)} = C_2(A, V) \|Eu\|_{H^1(\mathbb{R}^N)} \\ &\leq C(\Omega, V) C_2(A, V) \|u\|_{H^1(\Omega)} \leq C(\Omega, V) C_2(A, V) C_1^{-1}(A, \Omega) \|u\|_{H_A^1(\Omega)}, \end{aligned}$$

which concludes the proof. \square

We can now prove the following result:

Lemma 2.3. *Let $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be locally bounded. Let $u \in H_A^1(\Omega)$ and $\rho \in L^1(\mathbb{R}^N)$ with $\rho \geq 0$. Then*

$$\int_{\Omega} \int_{\Omega} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^2}{|x-y|^2} \rho(x-y) dx dy \leq C \|\rho\|_{L^1} \|u\|_{H_A^1(\Omega)}^2$$

where C depends only on Ω and A .

Proof. Let $V \subset \mathbb{R}^N$ be a fixed compact set with $\Omega \Subset V$. Given $u \in H_A^1(\Omega)$, by Lemma 2.2, there exists a function $\tilde{u} \in H_A^1(\mathbb{R}^N)$ with $\tilde{u} = u$ on Ω and $\tilde{u} = 0$ on V^c . By Lemma 2.1 and 2.2,

$$(2.1) \quad \int_{\mathbb{R}^N} |\tilde{u}(y+h) - e^{ih \cdot A(y+\frac{h}{2})} \tilde{u}(y)|^2 dy \leq C|h|^2 \|\tilde{u}\|_{H_A^1(\mathbb{R}^N)}^2 \leq C|h|^2 \|u\|_{H_A^1(\Omega)}^2,$$

for some positive constant C depending on Ω and A . Then, in light of (2.1), we get

$$\begin{aligned} \int_{\Omega} \int_{\Omega} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^2}{|x-y|^2} \rho(x-y) dx dy &\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \rho(h) \frac{|\tilde{u}(y+h) - e^{ih \cdot A(y+\frac{h}{2})} \tilde{u}(y)|^2}{|h|^2} dy dh \\ &= \int_{\mathbb{R}^N} \frac{\rho(h)}{|h|^2} \left(\int_{\mathbb{R}^N} |\tilde{u}(y+h) - e^{ih \cdot A(y+\frac{h}{2})} \tilde{u}(y)|^2 dy \right) dh \\ &\leq C \|\rho\|_{L^1} \|u\|_{H_A^1(\Omega)}^2, \end{aligned}$$

which concludes the proof. \square

Lemma 2.4. *Let $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be locally bounded and let $u \in H_{0,A}^1(\Omega)$. Then, we have*

$$(1-s) \int_{\mathbb{R}^{2N}} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^2}{|x-y|^{N+2s}} dx dy \leq C \|u\|_{H_A^1(\Omega)}^2$$

where C depends only on Ω and A .

Proof. Given $u \in C_c^\infty(\Omega)$, by Lemma 2.1 we have

$$\int_{\mathbb{R}^N} |u(y+h) - e^{ih \cdot A(y+\frac{h}{2})} u(y)|^2 dy \leq C |h|^2 \|u\|_{H_A^1(\Omega)}^2,$$

for some $C > 0$ depending on Ω and A and all $h \in \mathbb{R}^N$ with $|h| \leq 1$. Then, we get

$$\begin{aligned} (1-s) \int_{\mathbb{R}^{2N}} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^2}{|x-y|^{N+2s}} dx dy &\leq (1-s) \int_{\mathbb{R}^{2N}} \frac{|u(y+h) - e^{ih \cdot A(y+\frac{h}{2})} u(y)|^2}{|h|^{N+2s}} dy dh \\ &= (1-s) \int_{\{|h| \leq 1\}} \frac{1}{|h|^{N+2s}} \left(\int_{\mathbb{R}^N} |u(y+h) - e^{ih \cdot A(y+\frac{h}{2})} u(y)|^2 dy \right) dh \\ &\quad + 4(1-s) \int_{\{|h| \geq 1\}} \frac{1}{|h|^{N+2s}} dh \|u\|_{L^2(\Omega)}^2 \\ &\leq (1-s) \int_{\{|h| \leq 1\}} \frac{1}{|h|^{N+2s-2}} dh \|u\|_{H_A^1(\Omega)}^2 + C \|u\|_{L^2}^2 \leq C \|u\|_{H_A^1(\Omega)}^2. \end{aligned}$$

The assertion then follows by a density argument. \square

If $A|_{\Omega}$ is smooth (and extended if necessary to a locally bounded field on Ω^c), we get the following result.

Theorem 2.5. *Assume that $A \in C^2(\bar{\Omega})$. Let $u \in H_A^1(\Omega)$ and consider a sequence $\{\rho_n\}_{n \in \mathbb{N}}$ of nonnegative radial functions in $L^1(\mathbb{R}^N)$ with*

$$(2.2) \quad \lim_{n \rightarrow \infty} \int_0^\infty \rho_n(r) r^{N-1} dr = 1,$$

and such that, for every $\delta > 0$,

$$(2.3) \quad \lim_{n \rightarrow \infty} \int_\delta^\infty \rho_n(r) r^{N-1} dr = 0$$

Then, we have

$$(2.4) \quad \lim_{n \rightarrow \infty} \int_{\Omega} \int_{\Omega} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^2}{|x-y|^2} \rho_n(x-y) dx dy = 2K_N \int_{\Omega} |\nabla u - iA(x)u|^2 dx$$

being K_N the constant introduced in (1.1).

Proof. Let us first observe that by (2.2) and (2.3) we easily obtain that, for every $\delta > 0$,

$$(2.5) \quad \lim_{n \rightarrow \infty} \int_0^\delta \rho_n(r) r^N dr = \lim_{n \rightarrow \infty} \int_0^\delta \rho_n(r) r^{N+1} dr = 0.$$

In fact, taken any $0 < \tau < \delta$, we have

$$\int_0^\delta \rho_n(r) r^N dr = \int_0^\tau \rho_n(r) r^N dr + \int_\tau^\delta \rho_n(r) r^N dr \leq \tau \int_0^\tau \rho_n(r) r^{N-1} dr + \delta \int_\tau^\infty \rho_n(r) r^{N-1} dr,$$

from which formula (2.5) follows using (2.2), (2.3) and letting $\tau \searrow 0$. We follow the main lines of the proof in [3]. Setting

$$F_n^u(x, y) := \frac{u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)}{|x-y|} \rho_n^{1/2}(x-y), \quad x, y \in \Omega, \quad n \in \mathbb{N},$$

by virtue of Lemma 2.3, for all $u, v \in H_A^1(\Omega)$, recalling (2.2) we have

$$\left| \|F_n^u\|_{L^2(\Omega \times \Omega)} - \|F_n^v\|_{L^2(\Omega \times \Omega)} \right| \leq \|F_n^u - F_n^v\|_{L^2(\Omega \times \Omega)} \leq C \|u - v\|_{H_A^1(\Omega)},$$

for some $C > 0$ depending on Ω and A . This allows to prove (2.4) for $u \in C^2(\bar{\Omega})$. If we set

$$\varphi(y) := e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y),$$

since

$$\nabla_y \varphi(y) = e^{i(x-y) \cdot A(\frac{x+y}{2})} \left(\nabla_y u(y) - iA\left(\frac{x+y}{2}\right) u(y) + \frac{i}{2} u(y) (x-y) \cdot \nabla_y A\left(\frac{x+y}{2}\right) \right),$$

if $x \in \Omega$, a second order Taylor expansion gives (since $u, A \in C^2$, then $\nabla_y^2 \varphi$ is bounded on $\bar{\Omega}$)

$$u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y) = \varphi(x) - \varphi(y) = (\nabla u(x) - iA(x)u(x)) \cdot (x-y) + \mathcal{O}(|x-y|^2).$$

Hence, for any fixed $x \in \Omega$,

$$(2.6) \quad \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|}{|x-y|} = \left| (\nabla u(x) - iA(x)u(x)) \cdot \frac{x-y}{|x-y|} \right| + \mathcal{O}(|x-y|).$$

Fix $x \in \Omega$. If we set $R_x := \text{dist}(x, \partial\Omega)$, integrating with respect to y , we have

$$(2.7) \quad \begin{aligned} \int_\Omega \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^2}{|x-y|^2} \rho_n(x-y) dy &= \int_{B(x, R_x)} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^2}{|x-y|^2} \rho_n(x-y) dy \\ &+ \int_{\Omega \setminus B(x, R_x)} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^2}{|x-y|^2} \rho_n(x-y) dy. \end{aligned}$$

The second integral goes to zero by conditions (2.3), since

$$\lim_{n \rightarrow \infty} \int_{\Omega \setminus B(x, R_x)} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^2}{|x-y|^2} \rho_n(x-y) dy \leq C \lim_{n \rightarrow \infty} \int_{B^c(0, R_x)} \rho_n(z) dz = 0.$$

Now, in light of (2.6), following [3] we compute

$$\begin{aligned} \int_{B(x, R_x)} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^2}{|x-y|^2} \rho_n(x-y) dy &= Q_N |\nabla u(x) - iA(x)u(x)|^2 \int_0^{R_x} r^{N-1} \rho_n(r) dr \\ &+ \mathcal{O} \left(\int_0^{R_x} r^N \rho_n(r) dr \right) + \mathcal{O} \left(\int_0^{R_x} r^{N+1} \rho_n(r) dr \right), \end{aligned}$$

where we have set

$$Q_N = \int_{\mathbb{S}^{N-1}} |\omega \cdot \mathbf{e}|^2 d\mathcal{H}^{N-1}(\omega),$$

being $\mathbf{e} \in \mathbb{R}^N$ a unit vector. Letting $n \rightarrow \infty$ in (2.7), the result follows by dominated convergence, taking into account formulas (2.5). \square

3. PROOFS OF THEOREM 1.1 AND 1.2

3.1. Proof of Theorem 1.1. If $r_\Omega := \text{diam}(\Omega)$, we consider a radial cut-off $\psi \in C_c^\infty(\mathbb{R}^N)$, $\psi(x) = \psi_0(|x|)$ with $\psi_0(t) = 1$ for $t < r_\Omega$ and $\psi_0(t) = 0$ for $t > 2r_\Omega$. Then, by construction, $\psi_0(|x - y|) = 1$, for every $x, y \in \Omega$. Furthermore, let $\{s_n\}_{n \in \mathbb{N}} \subset (0, 1)$ be a sequence with $s_n \nearrow 1$ as $n \rightarrow \infty$ and consider the sequence of radial functions in $L^1(\mathbb{R}^N)$

$$(3.1) \quad \rho_n(|x|) = \frac{2(1-s_n)}{|x|^{N+2s_n-2}} \psi_0(|x|), \quad x \in \mathbb{R}^N, \quad n \in \mathbb{N}.$$

Notice that (2.2) holds, since

$$\lim_{n \rightarrow \infty} \int_0^{r_\Omega} \rho_n(r) r^{N-1} dr = \lim_{n \rightarrow \infty} 2(1-s_n) \int_0^{r_\Omega} \frac{1}{r^{2s_n-1}} dr = \lim_{n \rightarrow \infty} r_\Omega^{2-2s_n} = 1,$$

and

$$\lim_{n \rightarrow \infty} \int_{r_\Omega}^{2r_\Omega} \rho_n(r) r^{N-1} dr = \lim_{n \rightarrow \infty} 2(1-s_n) \int_{r_\Omega}^{2r_\Omega} \frac{\psi_0(r)}{r^{2s_n-1}} dr = 0.$$

In a similar fashion, for any $\delta > 0$, there holds

$$\lim_{n \rightarrow \infty} \int_\delta^\infty \rho_n(r) r^{N-1} dr \leq \lim_{n \rightarrow \infty} 2(1-s_n) \int_\delta^{2r_\Omega} \frac{1}{t^{2s_n-1}} dt = 0.$$

Then Theorem 1.1 follows directly from Theorem 2.5 using ρ_n as defined in (3.1). \square

3.2. Proof of Theorem 1.2. In light of Theorem 1.1 and since $u = 0$ on Ω^c , we have

$$\lim_{s \nearrow 1} (1-s) \int_{\mathbb{R}^{2N}} \frac{|u(x) - e^{i(x-y) \cdot A(\frac{x+y}{2})} u(y)|^2}{|x-y|^{N+2s}} dx dy = K_N \int_\Omega |\nabla u - iA(x)u|^2 dx + \lim_{s \nearrow 1} R_s,$$

where

$$R_s \leq 2(1-s) \int_\Omega \int_{\mathbb{R}^N \setminus \Omega} \frac{|u(x)|^2}{|x-y|^{N+2s}} dx dy.$$

On the other hand, arguing as in the proof of [5, Proposition 2.8], we get $R_s \rightarrow 0$ as $s \nearrow 1$ when $u \in C_c^\infty(\Omega)$ and, on account of Lemma 2.4, for general function in $H_{0,A}^1(\Omega)$ by a density argument. \square

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REFERENCES

- [1] G. Arioli, A. Szulkin, *A semilinear Schrödinger equation in the presence of a magnetic field*, Arch. Ration. Mech. Anal. **170** (2003), 277–295. [1](#)
- [2] J. Avron, I. Herbst, B. Simon, *Schrödinger operators with magnetic fields. I. General interactions*, Duke Math. J. **45** (1978), 847–883. [1](#)
- [3] J. Bourgain, H. Brezis, P. Mironescu, *Another look at Sobolev spaces*, in *Optimal Control and Partial Differential Equations. A Volume in Honor of Professor Alain Bensoussan's 60th Birthday* (eds. J. L. Menaldi, E. Rofman and A. Sulem), IOS Press, Amsterdam, 2001, 439–455. [1](#), [5](#)
- [4] J. Bourgain, H. Brezis, P. Mironescu, *Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications*, J. Anal. Math. **87** (2002), 77–101. [1](#)
- [5] L. Brasco, E. Parini, M. Squassina, *Stability of variational eigenvalues for the fractional p -Laplacian*, Discrete Contin. Dyn. Syst. A **36** (2016), 1813–1845. [6](#)
- [6] P. d'Avenia, M. Squassina, *Ground states for fractional magnetic operators*, <http://arxiv.org/abs/1601.04230> [1](#)
- [7] L.C. Evans, *Partial Differential Equations: Second Edition Graduate Series in Mathematics*, AMS, 2010. [3](#)
- [8] R.L. Frank, E.H. Lieb, R. Seiringer, *Hardy-Lieb-Thirring inequalities for fractional Schrödinger operators*, J. Amer. Math. Soc. **21** (2008), 925–950. [1](#)
- [9] T. Ichinose, *Magnetic relativistic Schrödinger operators and imaginary-time path integrals*, Mathematical physics, spectral theory and stochastic analysis, 247–297, Oper. Theory Adv. Appl. **232**, Birkhäuser/Springer, Basel, 2013. [1](#)

- [10] V. Maz'ya and T. Shaposhnikova, *On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces*, J. Funct. Anal. **195** (2002), 230–238. [1](#)
- [11] M. Reed, B. Simon, *Methods of modern mathematical physics, I, Functional analysis*, Academic Press, Inc., New York, 1980 [1](#)

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